HYDROMAGNETIC DYNAMO THEORY, I.

BY

WALTER M. ELSASSER

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Walter M. Elsasser

Department of Physics
University of Utah
Salt Lake City, Utah

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(To be concluded in a later report)

1. Outline of Problem.

One of the striking results of modern astrophysics is the ubiquity of magnetic fields in the universe. At present we know a large number of stars with magnetic fields of the order of several thousand gauss (Babcock and Cowling, 1953). A rather large fraction of them, if not all, have variable fields, so much so that one is tempted to consider variability as an intrinsic feature of magnetic stars. Moreover, there is evidence that the clouds of rarefied gas which are found in galactic space carry magnetic fields. The most successful theories of the origin of cosmic rays seem to be those that assume the acceleration of cosmic-ray particles to be caused by the mean action of these fields (Fermi, 1949; Morrison, Olbert, and Rossi, 1954).

The theory of hydromagnetism, or magnetohydrodynamics,* has only fairly recently been developed in an effort to explain these cosmic magnetic fields, and it bids fair to provide such an explanation. As we shall see (Sec. 4) small stray magnetic fields can be amplified by the action of suitable fluid motions. Now fluids of large dimensions are as a rule highly turbulent. One might therefore expect the velocity distribution of these fluids to follow some statistical pattern, so that the amplification of stray magnetic fields may be determined from statistical principles. The final result of such randomly distributed

*The words hydromagnetism and magnetohydrodynamics, and the corresponding adjectives have been used rather indiscriminately in the literature. We are using here the former term for aesthetic reasons and as being more economical of space. This usage has the approval of the distinguished Secretary of the American Physical Society.

amplificatory processes would then be a statistical equilibrium between the turbulent motion and the more or less irregular magnetic fields generated. This is the approach taken by Batchelor (1950) who succeeded in estimating the magnetic spectrum that would be in equilibrium with the velocity spectrum of turbulence. Such a picture gives us a first insight into the mechanism of the generation of magnetic fields under the conditions studied by the astrophysicist. It indicates in a provisional fashion at least, that the hydromagnetic theory can account for the presence of magnetic fields in the universe without ad hoc assumptions; the generation of magnetic fields of the order of magnitude observed follows without difficulty from the application of Maxwell's electromagnetic field equations to moving, electrically conducting fluids of large dimensions (Sec. 2).

From this viewpoint the older experiences regarding magnetic fields of the earth and the sun appear in a new light. They are taken out of their conceptual isolation and appear as special cases, relatively more accessible to our observation, of a universal phenomenon. The pertinent facts concerning the earth's interior which form the physical background for the earth's magnetic field have been reviewed in detail some years ago (Elsasser, 1950). We shall confine ourselves here to a few words. The earth has been shown from seismological observations to have a liquid core set off from its outer, solid part by a sharp surface of discontinuity. The radius of the core is about 3500 km, corresponding to 55% of the earth's radius. Geochemical evidence indicates that the material constituting the core is primarily molten iron with perhaps an admixture of nickel and

possibly some minor constituents in solution. In order to apply hydromagnetic theory to the earth's core it suffices to assume that the material is a good electrical conductor, its conductivity being comparable at least in order of magnitude to that of ordinary metals. Beyond this it is merely necessary to assume that the core is fluid and that internal motions occur. The geomagnetic secular variation puts the fluid character of the core in evidence. The secular variation may be analyzed into a spectrum whose prime components have periods of the order of a few hundred years. There is no known way of accounting for periods of this order on the basis of mechanical, thermal or other processes occuring in the solid outer parts of the earth, whereas the theory which assumes this secular variation to be associated with fluid motions in the earth's core is able to explain them quite satisfactorily (Elsasser, 1950). In the present review we shall not yet discuss these phenomena in detail (although they are well suited to compare the theory in a quantitative fashion with direct observations). Instead, we shall focus our attention on the basic hydromagnetic processes by which the earth's dipole field is generated and maintained. These processes are not necessarily the ones which are most directly revealed by the secular variation, they take place in the deeper parts of the earth's core, whereas the geomagnetic secular variation may be shown to inform us only about the conditions in a very shallow layer of fluid adjacent to the surface of the core.

The driving mechanism by which the fluid motion, and hence indirectly the magnetic field, is maintained is generally assumed to be thermal convection (Bullard, 1949) although convective

motion induced by other means, in particular progressive sedimentation (Urey, 1952) need not be rejected. In any event, the power supplied from thermal sources (radioactive heat in the core, plausible radial temperature gradients) is, even under very conservative assumptions, more than sufficient to maintain throughout the lifetime of the earth fluid motions of the magnitude inferred from the observed secular variation (0.1-1 mm/sec). The details of the primary driving mechanism need not concern us in this review. it will appear, however (Sec. 5) that the fluid motions must be essentially three-dimensional; a pattern restricted, e.g., to spherical sheets or to meridional planes is not adequate to produce dynamo action. Recent seismological research (Bullen, 1954) makes it likely that the central part of the core is again solid, but the volume of this inner, solid sphere is only a very small fraction of the entire volume of the core, and for the purposes of the analysis given below it will be sufficient to assume the earth's core as a homogeneous fluid sphere. Compressibility effects are not likely to be important and so the fluid core may be considered as incompressible; moreover the electrical conductivity will be assumed constant.

There is another class of extensively studied phenomena which can be attributed to hydromagnetic effects, namely, the magnetic fields observed on the sun, particularly in sunspots. Every sunspot has a magnetic field associated with it; the larger the spots, the larger as a rule the fields. The field strength in the larger spots goes up to a saturation value of about 3000 gauss with a margin of fluctuation of nearly $\pm$ 1000 gauss. Sunspots appear very frequently in pairs, the line connecting the two spots running in an east-westerly direction, along a circle

of latitude. The two members of a sunspot pair always have opposite magnetic polarity. The "leader" spot (the one appearing ahead in the sense of the solar rotation) always has one definite polarity during one and the same sunspot cycle, the "follower" having the opposite polarity. The leader is in the average larger and has a larger magnetic field than the follower; single spots as a rule have the same magnetic polarity as the leader spots. Sunspots appear and grow to their full size in the course of a few days; they then gradually decay during the course of a few weeks, some of them persisting over a few months. The solar latitude at which the spots are seen is a function of the 11 1/2-year sunspot cycle. In the beginning of the cycle the spots appear at a latitude of about 30°. As the sunspot cycle progresses the spots appear at lower and lower latitudes until towards the end of the cycle they are found very near the solar equator, at latitudes of 5-10°. At the same time new spots begin to appear around the latitude of 30°, but these spots have opposite magnetic polarities. During the entire subsequent cycle the polarity of the spots is the opposite of that found in the previous cycle; for the next 11 1/2-year cycle the polarity reverses again, and so on. Clearly the complete sunspot cycle must contain the reversal of the field, and extends over 23 years. There are many other observations that indicate the presence of magnetic fields in the sun other than sunspot fields and the general character of these fields appears to vary with the sunspot cycle, but these fields are very much smaller in magnitude than the sunspot fields. We may refer here to a very comprehensive recent work on the sun (Kuiper, ed. 1953).

Let us now return to our initial remark that magnetic fields in cosmic fluids may be produced by amplification from

small initial fields. We referred to turbulence as being able to generate such fields in a statistical manner. Now while the earth's field is highly irregular in the details, by far the largest part of the field has the form of a dipole roughly parallel to the earth's axis. The prominent feature of solar magnetism is the 23-year cycle which, while subject to fluctuations in strength and also to certain fluctuations in length, has been observed to occur with consistent regularity over the last 200 years, and there is no reason to doubt that it is a relatively stable feature of solar activity. Similarly, when stars are observed to have over-all fields of several thousand gauss, a systematic cause must be operative. It is clear that if the observed phenomena are to be explained by hydromagnetic amplification, some regularity of the pattern of fluid motion must underly them. Fluid motions which produce relatively stationary or periodic magnetic fields will be designated as hydromagnetic dynamos. Since fluid motions in large dimensions are turbulent, or at least more or less irregular, we shall not be concerned with rigorous solutions of the hydro-magnetic equations, but with typical solutions which demonstrate the stability of the fields in the mean. This point of view is in full agreement with the observations which show that none of the observed parameters of the field is rigorously constant; all of them are subject to certain rather appreciable margins of fluctuations. As an example of this we may mention that the earth's dipole moment has decreased by about 5% since 1850 (Elsasser, 1950) although other observational data leave no doubt that the earth must have possessed a magnetic dipole moment for a very long time indeed, so that the present decline is in all

likelihood only a temporary fluctuation. A disregard of these pronouncedly irregular fluctuations of the field has led to the so-called "fundamental" theories which try to relate geomagnetism and other cosmic magnetic phenomena to properties of matter in the large not contained in the conventional equations of classical physics. The hydromagnetic theory shows that classical physics can account for these phenomena, but that the particular parameters or combination of parameters have, for good and sufficient reason, escaped observation in the laboratory (Sec. 2).

The dynamo theory requires that the fluid motions exhibit certain regularities in order that magnetic fields may be maintained in the average. In other words, there must be some ordering principle that controls the fluid motions, and we must identify this principle. One's first idea would seem to be a search for arguments of symmetry: to find some symmetry requirement that restricts the generality of the fluid motions and impresses upon them a relatively simple pattern. But such a search proves to be in vain. There is every indication that an appreciable degree of symmetry of the fluid motions will suppress or cancel the effects of hydromagnetic amplification (Sec. 5); thus we are led to look for patterns of the fluid motion of a low rather than a high degree of symmetry. To make a long story short, the ordering principle which engenders the most conspicuous hydromagnetic effects may be identified as the Coriolis force acting upon the fluid motions in a rotating system. Such is the working hypothesis of this paper. The Coriolis deflection affects the fluid motions in such a fashion that the resulting pattern does not in general admit of any symmetry operations. Stationary or periodic

hydromagnetic amplification is thus related to the rotation of the fluid mass in which it occurs. There is some observational presumption in favor of this idea, since the stars with strong magnetic fields seem to rotate rather rapidly. The evidence is, however, not entirely conclusive and the assumption must be justified by working out its dynamical consequences, we might remark that we are not informed of any other dynamical principle which could be adduced to explain magnetic fields of the type observed.

2. Field Equations, Dimensions.

We shall use the rationalized mks system throughout. The electromagnetic fields will be assumed to obey Maxwell's equations, hence the material medium may at any point be described by the three constants σ, μ, ε. We shall assume μ and ε as constant throughout space and shall for simplicity assume σ constant for a given fluid, although it would not be difficult to generalize the theory to fluids with variable σ. We then have

$$\nabla \times \underset{\sim}{E} = -\partial \underset{\sim}{B}/\partial t \;, \quad \nabla \cdot \underset{\sim}{E} = \eta/\varepsilon \tag{2.1}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \;, \quad \nabla \times \underset{\sim}{B} = \mu \underset{\sim}{J} \tag{2.2}$$

where η, $\underset{\sim}{J}$ are charge and current density. Next we write down the most general expression for the current density admissible in Maxwell's theory,

$$\underset{\sim}{J} = \sigma \underset{\sim}{E} + \sigma \underset{\sim}{v} \times \underset{\sim}{B} + \varepsilon \partial \underset{\sim}{E}/\partial t + \eta v \tag{2.3}$$

where v is the material velocity of the fluid and where the terms on the right represent, respectively, the conduction current, the induction current, the displacement current, and the convection current. Formula (2.3) differs from the conventional expression for the total current by the second and fourth terms on the right which contain the fluid velocity, $\underset{\sim}{v}$. For the detailed derivation of these terms any extensive text on electrodynamics may be consulted.

We shall now proceed to show that the last two terms of (2.3) are negligibly small under the conditions met with in cosmic fluids. We shall use braces, $\{\ \}$, to designate the order of magnitude of a given physical quantity; let in particular $\{\lambda\}$

stand for a typical length and $\{\omega\}$ for a typical reciprocal time. We first note that

$$\{v/c\} = \{\beta\} \ll 1 \tag{2.4}$$

since the velocities of cosmic fluids rarely exceed a few km/sec. We now compare the displacement current, the third term on the right-hand side of (2.3), to the conduction current, the first term. The ratio is

$$\{\omega\varepsilon/\sigma\} = \{\Upsilon\} \tag{2.5}$$

where Υ is exceedingly small. To show this, let $\sigma = 10^7$, the conductivity of ordinary iron; then for $\Upsilon = 1$ we find $\omega \approx 10^{18}$. This shows that for the frequencies of all macroscopic motions Υ is utterly negligible. (The quantity (2.5) is familiar to the student of metal optics where it is used in the same way as here, namely, to measure the ratio of displacement to conduction current.) It is readily shown that the ratio of convection current to conduction current is also given by (2.5). From (2.1) we have indeed $\{\eta\} = \{\varepsilon E/\lambda\}$, hence this ratio is

$$\{\eta v/\sigma E\} = \{\varepsilon v/\sigma\lambda\} = \{\Upsilon\} \tag{2.6}$$

if we identify $\omega = v/\lambda$ as a typical frequency of the material motion of the fluid.

We next find for the ratio of the electrical to the magnetic field energy, using (2.1)

$$\{\varepsilon E^2/\mu^{-1}B^2\} = \{E^2/c^2B^2\} = \{\lambda^2\omega^2/c^2\} = \{\beta^2\} \tag{2.7}$$

if we again identify $\lambda\omega$ with the velocity of the fluid, as is proper in an entirely Maxwellian scheme. Hence the electrostatic field energy is small and it follows from familiar arguments that in our approximation all electromagnetic processes are aperiodic.

These estimates call for some further comments. We see from the numerical estimate of γ that even for a moderately ionized gas the electromagnetic phenomena are aperiodic for frequencies in the radio spectrum. Thus plasma oscillations at these frequencies require that the description in terms of the macroscopic equations of Maxwell's theory be invalid; this is because (2.7) becomes invalid for the velocity of the electronic component of the plasma. In the present article we are dealing, however, only with the macroscopic, average motions of the conducting material; the characteristic frequencies are then lower by many powers of ten than the frequencies of the radio spectrum; hence (2.5), (2.6) and (2.7) are certainly small and our approximation may safely be applied. Now (2.2) and (2.3) give

$$\nabla \times \underset{\sim}{B} = \mu\sigma \underset{\sim}{E} + \mu\sigma \underset{\sim}{v} \times \underset{\sim}{B} \qquad (2.8)$$

Continuing our dimensional analysis (Elsasser, 1954) we compare the order of magnitude of the three terms in (2.8). We first notice that by virtue of (2.7) the two terms on the right hand side are of comparable order of magnitude. The ratio of any one of these terms to the net current on the left is

$$\left\{ \mu\sigma\lambda v \right\} = \left\{ R_m \right\} \qquad (2.9)$$

where the non-dimensional quantity R_m will be designated as the <u>magnetic Reynolds number</u>. If we substitute numerical values for the quantities on the left of (2.9) we find that R_m is numerically large for cosmic fluids. It is of the order of magnitude of several hundred to perhaps a thousand for the earth's metallic core, depending on the detailed assumptions made, and is of order 10^5 or more for most astrophysical conditions. This constitutes the

essential difference of cosmic hydromagnetism from laboratory conditions where, as one readily verifies, R_m is numerically small. Hence

$$\mathbf{E} \sim -\mathbf{v} \times \mathbf{B} \tag{2.10}$$

and the mechanism whereby the magnetic field is maintained is quite at variance from the conventional situation where the net current is the source of $\mathbf{B}$.

Operating with the curl on (2.8) and using the field equations we eliminate $\mathbf{E}$ and find

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nu_m \nabla^2 \mathbf{B} \tag{2.11}$$

where we have written

$$\nu_m = (\mu\sigma)^{-1} \tag{2.12}$$

The quantity ν_m will be designated as the magnetic viscosity. We see from (2.9) that R_m differs from the conventional hydrodynamic Reynolds number, R, only in that ν_m replaces the kinematical viscosity, ν, of the fluid.

The physical implications of (2.11) are best brought out in terms of an integral equation. To obtain it we integrate (2.1) along a contour C and use (2.8), then by Stokes' theorem

$$(\partial/\partial t)\int \mathbf{B}_n d\sigma = -\int \mathbf{E} \cdot d\mathbf{C}$$

$$= \int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{C} - \nu_m \int (\nabla \times \mathbf{B}) \cdot d\mathbf{C}$$

Now if the first integrand on the right-hand side is written $\mathbf{B} \cdot (d\mathbf{C} \times \mathbf{v})$, the integral can be given a simple geometrical meaning: it becomes $-\int B_n d\sigma$ where the integration extends over the strip that the contour C subtends in its motion during the time dt. Since $\int B_n d\sigma = 0$ for any closed surface, we can write

this

$$(d/dt)\int \underset{\sim}{B}_n d\sigma = -\nu_m \int (\nabla \times \underset{\sim}{B}) \cdot d\underset{\sim}{C} \qquad (2.13)$$

On the left there appears now the substantial derivative, referring to motion with the fluid particles. Clearly, if (2.13) is applied to any arbitrary contour, it is equivalent to the differential equation (2.11).

Next take the ratio of the left-hand side of (2.13) to the right-hand side. This ratio is readily seen from (2.12) and (2.9) to be just $\{R_m\}$. Hence under geophysical and astrophysical conditions we have very approximately

$$(d/dt)\int \underset{\sim}{B}_n d\sigma = 0 \qquad (2.14)$$

which is usually enunciated by stating that the magnetic lines of force are carried along bodily with the fluid; they are "frozen" as it were, in the conducting fluid. On applying a well-known vector identity to the first term on the right of (2.11) we obtain the differential equation in the form

$$d\underset{\sim}{B}/dt = (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{v} - \underset{\sim}{B}(\nabla \cdot \underset{\sim}{v}) + \nu_m \nabla^2 \underset{\sim}{B}$$

which may be further simplified on introducing from the equation of continuity

$$\nabla \cdot \underset{\sim}{v} = \rho\, d(\rho^{-1})/dt$$

with the result

$$d(\rho^{-1}\underset{\sim}{B})/dt = (\rho^{-1}\underset{\sim}{B} \cdot \nabla)\underset{\sim}{v} + \rho^{-1}\nu_m \nabla^2 \underset{\sim}{B} \qquad (2.15)$$

an equation that exhibits more clearly the role of compressibility (Truesdell, 1950).

This equation is remarkable in that it shows a complete formal analogy to the Helmholtz theorem of the conservation of vorticity. Indeed, if we replace $\underset{\sim}{B}$ by the vorticity vector and

ν_m by ν, (2.15) becomes just the general vorticity-conservation theorem. The physical implications of this result for the tubes of vorticity, which now may be transferred at once to the tubes of magnetic flux, are well enough known and may be found in almost any text on hydrodynamics. There is one point, however, where a great deal of misunderstanding appears to exist in the literature: One has become habituated to saying that not only must flux be conserved during the motion (for vanishing ν or ν_m) but that also the lines of vorticity or of the magnetic field must be closed in the absence of sources. From the existing discussions one is often led to the implicit belief that the condition $\nabla \cdot B = 0$ requires that the magnetic lines of force must be closed curves. This statement is certainly incorrect. The subject has recently been studied in some detail by McDonald (1954). He shows that there are two conditions under which the field lines are not closed. In the first place they can terminate in singularities that is points, lines, or surfaces where $B = 0$. Examples of ordinary current configurations where such singularities appear can be constructed in abundance; similarly such singularities may be present in problems of hydromagnetism. A second class of non-closed lines is that of lines which are "ergodic", that is cover a region everywhere densely. As an example for the latter, consider a current system consisting of a straight current-carrying wire and a second wire forming a circular loop which lies in a plane normal to the straight wire with its center on the latter. The lines of force in the neighborhood of the loop are clearly spirals; a line going through a given point will be closed only when the ratio of the currents flowing in the two wires has

certain rational values, otherwise the line will fill a torus-shaped surface everywhere densely. It cannot be our aim to consider in detail these somewhat involved properties of field lines which pertain to analytical vector geometry and are of more interest perhaps to the topologist than to the physicist. One may readily perceive, even without delving into more rigorous mathematics, that the closed lines of force form, set-theoretically speaking, only a subset of measure zero of the set of all possible field lines.

One thing stands out clearly: "Intuitive" arguments regarding the existence of closed lines of force and the impossibility of generating new closed lines by deformation of the fluid are of no value. They cannot be used as arguments in a discussion of hydromagnetic amplification unless they can be converted into formulas based on the vector-field equations of the theory. Much unjustified scepticism against the reality of hydromagnetic processes has arisen from such intuitive reasoning. It can be avoided only on abandoning the line-of-force concept of elementary textbooks in favor of the theory, at the same time more rigorous and more simple, of vector densities or fluxes, which is implied by the field equations.

Returning now to the equation (2.11) we see readily that the ratio of the first to the second term on the right-hand side is again $\{R_m\}$. This is completely analogous to conventional hydrodynamics where the ratio of the dynamical to the frictional terms is $\{R\}$. There is this difference in practice, namely, that we can on occasion realize fairly high values of R in the laboratory, whereas with conventional materials R_m remains

small; hence while we are familiar with turbulence, we are not well acquainted with the type of phenomena characteristic of cosmic hydromagnetism. Noting that (2.11) has the dimension $\{\omega B\}$ we may write the ratio of the two terms on the right

$$\{R_m\} = \{v/\lambda\omega_d\} = \{\omega_v/\omega_d\} \tag{2.16}$$

where ω_d is a characteristic frequency of the free decay of the field in the absence of motion and ω_v is characteristic of the fluid motions. We see here the physical basis of a dynamo theory: for sufficiently large R_m the magnetic field can be deformed (amplified) by the fluid motion before it has had time to decay. It remains for the dynamo theory to show that the deformation can occur in a sufficiently ordered fashion so that a mean magnetic field can survive.

Next, consider the mechanical motion of the fluid. From conventional electromagnetic theory we have for the density of the ponderomotive force of the magnetic field

$$\underline{F} = \underline{J} \times \underline{B} = \mu^{-1}(\nabla \times \underline{B}) \times \underline{B} \tag{2.17}$$

The corresponding forces of the electrostatic field are small by the same arguments as before. The force (2.17) will appear in the Stokes-Navier equations for the fluid motion. Using the well-known vector identity

$$(\nabla \times \underline{B}) \times \underline{B} = (\underline{B} \cdot \nabla)\underline{B} - \tfrac{1}{2}\nabla(\underline{B}^2) \tag{2.18}$$

we shall write these equations in the form

$$\partial \underline{v}/\partial t + (\underline{v} \cdot \nabla)\underline{v} = -\nabla\psi + (\rho\mu)^{-1}(\underline{B} \cdot \nabla)\underline{B} + \nu\nabla^2\underline{v} \tag{2.19}$$

where we have set

$$\rho\psi = p + U + (2\mu)^{-1}\underline{B}^2 \tag{2.20}$$

U being the gravitational potential. The equations (2.11) and (2.19) together with such subsidiary conditions as an equation of state, determine the dynamics of a cosmic fluid in which magnetic fields are present.

In the case of incompressibility, $\nabla \cdot \underset{\sim}{v} = 0$, this system admits of a remarkable symmetrization which puts in evidence the analogous roles of $\underset{\sim}{B}$ and $\underset{\sim}{v}$ in the theory. Letting

$$\underset{\sim}{P} = \underset{\sim}{v} + (\rho\mu)^{-\frac{1}{2}}\underset{\sim}{B}\,, \quad \underset{\sim}{Q} = \underset{\sim}{v} - (\rho\mu)^{-\frac{1}{2}}\underset{\sim}{B}$$

$$2\nu_1 = \nu + \nu_m \quad , \quad 2\nu_2 = \nu - \nu_m$$

we obtain, on adding or subtracting (2.11) and (2.19)

$$\begin{aligned} \partial \underset{\sim}{P}/\partial t + (\underset{\sim}{Q} \cdot \nabla)\underset{\sim}{P} &= -\nabla\psi + \nabla^2(\nu_1\underset{\sim}{P} + \nu_2\underset{\sim}{Q}) \\ \partial \underset{\sim}{Q}/\partial t + (\underset{\sim}{P} \cdot \nabla)\underset{\sim}{Q} &= -\nabla\psi + \nabla^2(\nu_1\underset{\sim}{Q} + \nu_2\underset{\sim}{P}) \end{aligned} \qquad (2.21)$$

where now (2.20) becomes

$$\psi = \rho^{-1}(p + U) + (\underset{\sim}{P} - \underset{\sim}{Q})^2/8 \qquad (2.22)$$

These equations were derived by Lundquist (1952) independently of the author (1950a). The symmetry of these equations might be somewhat misleading: thus we notice that ν_2 becomes negative when electromagnetic dissipation outweighs frictional dissipation, a fact that has no analog in ordinary hydrodynamics. The ratio

$$\nu/\nu_m = R_m/R = \mu\sigma\nu \qquad (2.23)$$

may be estimated from elementary kinetic theory for an ionized gas such as hydrogen (Elsasser, 1954). One finds the numerical value

$$\mu\sigma\nu = 2\cdot 10^{-4}\alpha/\rho$$

where α is the degree of ionization, ρ the density in mks units. This shows that electromagnetic dissipation outweighs frictional

loss in the interior of the stars where ρ is moderately large, whereas in the rarefied intragalactic medium, and also in regions such as the solar chromosphere or corona the electromagnetic loss is negligible compared to the loss by mechanical viscosity.

We next turn to the conservation laws. It may readily be shown that neither energy nor vorticity is conserved for the fluid motion alone. The work done by the fluid is just the negative of the work done by the ponderomotive force (2.17). The power delivered per unit volume is thus

$$-\underset{\sim}{v} \cdot \underset{\sim}{F} = \mu^{-1} \underset{\sim}{v} \times \underset{\sim}{B} \cdot (\nabla \times \underset{\sim}{B}) \tag{2.24}$$

We may of course obtain the same expression from the magnetic field equations. On scalar multiplication of (2.11) with $\underset{\sim}{B}$ and transformation of the first term on the right by a well-known vector identity we have, neglecting dissipation

$$(2\mu)^{-1} \partial \underset{\sim}{B}^2 / \partial t = \mu^{-1} \nabla \cdot \left[(\underset{\sim}{v} \times \underset{\sim}{B}) \times \underset{\sim}{B} \right] + \mu^{-1} (\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \nabla \times \underset{\sim}{B} \tag{2.25}$$

On integrating over a volume, the first term may be converted into a surface integral and can be made to vanish if the surface is extended to a region where $\underset{\sim}{v} = 0$ (this term represents essentially the Poynting flux, as may be seen by substituting (2.10)). The last term of (2.25) is identical with (2.24).

The curl of $\underset{\sim}{F}$ does not in general vanish and so there is transfer of vorticity between the fluid and the field. This is most conveniently expressed in terms of the Kelvin circulation theorem of hydrodynamics. Integrating (2.19) along a closed contour and again leaving out the frictional term we obtain

$$\begin{aligned} \frac{d}{dt} \int \rho \underset{\sim}{v} \cdot d\underset{\sim}{C} &= \mu^{-1} \int d\underset{\sim}{C} \cdot (\underset{\sim}{B} \cdot \nabla) \underset{\sim}{B} \\ &= \mu^{-1} \int (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} \cdot d\underset{\sim}{C} \end{aligned} \tag{2.26}$$

It is difficult to simplify the right-hand side further, but we shall not be required in the sequel to make explicit use of the theorem. While all the preceding integral theorems are of some interest, the one most significant in practice is the conservation of magnetic flux in the limit of small dissipation, given by (2.14). We shall make ample use of it later in the application to the dynamo theories; for further analysis and illustrative examples we may also refer the reader to a comprehensive treatment by Lundquist (1952).

It is possible to derive the field vectors from a vector potential, though the relationships are slightly different from those of more conventional electrodynamics. If we set, as usual

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\partial \mathbf{A}/\partial t \tag{2.27}$$

the first equations (2.1) and (2.2) are identically fulfilled. The second of (2.2) or, rather, (2.8) gives

$$\partial \mathbf{A}/\partial t = \mathbf{v} \times (\nabla \times \mathbf{A}) + \nu_m \nabla^2 \mathbf{A} \tag{2.28}$$

We cannot, however, set $\nabla \cdot \mathbf{A} = 0$, since the divergence of the first term on the right of (2.28) does not vanish (see Sec. 3). We now assume R_m large and neglect terms of order R_m^{-1}. The last term of (2.28) being of this order, we find

$$\partial/\partial t(\nabla \cdot \mathbf{A}) = \nabla \cdot [\mathbf{v} \times (\nabla \times \mathbf{A})] \tag{2.29}$$

By virtue of (2.10) which also holds apart from terms of order R_m^{-1}, (2.29) is identically fulfilled. Finally, η is determined from the second of (2.1). Thus the assumptions (2.27) are justified in the approximation in which electromagnetic dissipation may be neglected.

In most applications it is possible to ignore the

longitudinal (irrotational) part of $\underset{\sim}{E}$, and hence of $\underset{\sim}{A}$, altogether, since by the first of (2.1) it does not give rise to a magnetic field. If a method is given whereby $\underset{\sim}{A}$ may be split consistently into its transverse (divergence-free) and its longitudinal part (e.g. on using a system of normal modes, Sec. 4) we may re-introduce the condition $\nabla \cdot \underset{\sim}{A} = 0$ to supplement (2.27). Since, however, the divergence of the first term on the right of (2.28) does not in general vanish (see equ. 3.5) we must then supplement (2.28) by the condition that only its transverse component will be taken into account. Since the decomposition of a vector into its transverse and longitudinal parts is linear, this can often be done with comparative ease.

In the applications to astrophysical problems both R_m and R are numerically large. This means not only that we have turbulence but also that there will be an entire hierarchy of eddies, the larger eddies feeding energy into the smaller ones, according to the usual turbulence theory. The largest eddies correspond to the largest values of R possible, the smallest eddies correspond to either $R \sim 1$, or $R_m \sim 1$, depending on whichever one of these two numbers is the larger. If there is intense transfer of energy between the mechanical and electromagnetic degrees of freedom, at least among the smaller eddies (Batchelor, 1950) the cutoff of the turbulence spectrum must be determined by whatever mechanism of molecular dissipation is the more effective, mechanical friction or Joule's heat.

In a turbulent fluid the transport of physical properties such as heat content, momentum, vorticity, and so on, is determined by the corresponding molecular coefficients; the same

applies to thermal dissipation. Thus we shall be forced to replace ν_m by an eddy magnetic viscosity, ν_m' say, or else σ by an eddy electric conductivity, σ'. It is well known, of course, that these quantities are actually functions of the turbulent state.

To exhibit more formally the magnetic eddy stresses and the magnetic eddy diffusivity, we remember that purely mechanical stresses can be expressed as the divergence of a stress tensor (see for instance Sommerfeld, 1950). We shall assume in the remainder of this section that the fluid is incompressible. We shall use tensor notation in cartesian coordinates. The molecular viscous stresses are the divergence of the tensor, $\nu(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$. Again, in a turbulent medium we have the Reynolds stresses which are derived as follows: The Euler equations of the fluid are

$$\frac{\partial v_i}{\partial t} + \sum_k v_k \frac{\partial v_i}{\partial x_k} = - \frac{\partial \psi}{\partial x_i} \tag{2.30}$$

Now on account of incompressibility we have

$$\sum_k \frac{\partial}{\partial x_k}(v_i v_k) = \sum_k \frac{\partial v_i}{\partial x_k} v_k + v_i \sum_k \frac{\partial v_k}{\partial x_k} = \sum_k v_k \frac{\partial v_i}{\partial x_k}$$

We now set

$$v_i = v_i^o + v_i^1 \tag{2.31}$$

where v^o refers to the "smooth" and v^1 to the "turbulent" component of the velocity. We choose the decomposition (2.31) so that, on averaging over the irregular motion, $\overline{v_i^1} = 0$, hence $\overline{v_i} = v_i^o$, and

$$\overline{v_i v_k} = v_i^o v_k^o + \overline{v_i^1 v_k^1} = v_i^o v_k^o - S_{ik}$$

where S_{ik} is the Reynolds stress tensor. If we insert (2.31)

into (2.30) and average we obtain

$$\frac{\partial v_i^{\,o}}{\partial t} + \sum_k v_k^{\,o} \frac{\partial v_i^{\,o}}{\partial x_k} = - \frac{\partial \psi}{\partial x_k} + \sum_k \frac{\partial S_{ik}}{\partial x_k}$$

Now the ponderomotive force (2.17) may be expressed as the divergence of a Maxwellian stress tensor (see for instance Stratton, 1941). Referring to unit mass, this relation is

$$\rho^{-1} F_i = \sum_k \frac{\partial T_{ik}}{\partial x_k}, \quad T_{ik} = \frac{1}{\rho\mu}(B_i B_k - B^2 \delta_{ik}) \tag{2.32}$$

where δ_{ik} is the usual Kronecker symbol. We now split B_i into

$$B_i = B_i^{\,o} + B_i^{\,\prime} \tag{2.33}$$

where the same conditions for the averages hold as for (2.31). The equations of motion become

$$\frac{\partial v_i^{\,o}}{\partial t} + \sum_k v_k^{\,o} \frac{\partial v_i^{\,o}}{\partial x_k} = - \frac{\partial \psi}{\partial x_i} + \frac{1}{\rho\mu} \sum_k B_k^{\,o} \frac{\partial B_i^{\,o}}{\partial x_k}$$

$$+ \sum_k \frac{\partial}{\partial x_k}(S_{ik} + T_{ik}) \tag{2.34}$$

where in place of (2.20) we have now $\rho\psi' = p + U$, the term with $\underset{\sim}{B}^2$ having been absorbed into the stresses. We see that the mechanical stresses produced by the turbulent component of the magnetic field are exactly analogous to the purely mechanical stresses of turbulence.

A similar transformation may be effected for the magnetic field equations (2.11). We note in the first place that

$$\left\{ \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \right\}_i = \sum_k \frac{\partial}{\partial x_k}(v_i B_k - v_k B_i) \tag{2.35}$$

The parenthesis on the right represents an antisymmetrical tensor (it being well known that any vectorial product may be written

as such a tensor). This feature distinguishes the field equations (2.11) essentially from the equations of motion in which the stress tensors, mechanical as well as electromagnetic, are symmetrical. We introduce now the decompositions (2.31) and (2.33) and let

$$\overline{v_i B_k - v_k B_i} = v_i^{\circ} B_k^{\circ} - v_k^{\circ} B_i^{\circ} + I_{ik}$$

where I is the antisymmetrical turbulent induction tensor. Now (2.11) becomes, on omitting the term in v_m

$$\frac{\partial B_i^{\circ}}{\partial t} = \sum_k \frac{\partial}{\partial x_k}(v_i^{\circ} B_k^{\circ} - v_k^{\circ} B_i^{\circ}) + \sum_k \frac{\partial I_{ik}}{\partial x_k} \qquad (2.36)$$

where the last term now describes the turbulent diffusion of the magnetic field. It must be emphasized that in spite of the superficial similarity to the last term in (2.34) there is the fundamental difference that the induction tensor is antisymmetrical. In any one situation we can reverse the sense of energy transfer between fluid and field by merely reversing the direction of $\underset{\sim}{v}$, as is apparent from (2.24). Similarly, the sign of the components of I depends on correlations between the components of $\underset{\sim}{v}$ and of $\underset{\sim}{B}$ and the sign of these correlations may be reversed in the same way. Whereas the stress tensors S and T in (2.11) act in a way quite analogous to molecular viscous stresses, giving rise to irreversible effects only, this cannot be said of the tensor I: the classical proof of the irreversiblity of viscous stresses depends on the symmetry of the stress tensor. For stationary isotropic turbulence only the diagonal elements of the tensors remain, so that I vanishes in this case. This of course does not mean that there is no transfer, but that the transfers in the opposite direction balance. We can, however, have a systematic transfer,

and of either sign, if the turbulent pattern is sufficiently anisotropic. This question is closely related to that of eddies producing feedback in dynamo models which we shall take up in Sec. 7.

We may inquire into the comparative magnitude of the tensors S, T, I. A number of authors on astrophysical electrodynamics have assumed that equipartition holds for the energy, at least in order of magnitude:

$$\{\rho v^2\} = \{\mu^{-1}B^2\} \tag{2.37}$$

In this case it is readily seen from (2.32) that $\{T\} = \{v^2\} = \{S\}$. The magnitude of I is related to the R_m of the effective eddies in the same way in which S and T are related to R. As Batchelor (1950) points out, the largest eddies are driven mechanically and their energy is degraded before they can create an equilibrium magnetic field of their own dimensions and of magnitude (2.37); for the eddies of smaller dimensions but above cutoff we may assume such equilibrium to prevail. We would then be able to estimate that the ratio of the eddy-stress and eddy-diffusion terms to the dynamical terms is of comparable order in (2.34) and (2.36). There is, however, no basic need to assume that the equipartition (2.37) holds even approximately in a rotating system. A turbulent regime is not a statistical equilibrium but a dissipative process, at the best a stationary one. For an equilibrium the existence of detailed balancing is a sufficient (though not always a necessary) condition for equipartition. For non-equilibria the deviations from the equilibrium statistical distribution (equipartition) are larger, the more the system deviates from detailed balancing. In a rotating system the Coriolis force lacks mechanical reciprocity and hence

the system would not even permit of detailed balancing in a hypothetical equilibrium. Since the Coriolis force is essential for the dynamo mechanism which maintains the magnetic fields, it is preferable not to have recourse to the equipartition assumption (2.37) but to tackle the problem from basic dynamical principles, which is what we shall do later on.

3. Electric Fields, Potentials.

We shall now investigate more closely the range of validity of our equations. The electromagnetic field equations used were those of the conventional Maxwellian electrodynamics of ponderable bodies. There are certain observed cosmic phenomena which cannot be described on this basis, in particular radio noise and the acceleration of cosmic rays, but outside of these there exists a strong presumption that these macroscopic laws should hold for the slow motions and the large dimensions of the fluids considered. As before, we shall assume that β is numerically small and R_m numerically large. A question which immediately poses itself is that of the magnitude of the electrical effects associated with hydromagnetic phenomena, and this will now be considered.

It is well known that the electromagnetic field equations of ponderable bodies can be written in a relativistically invariant form (Minkowski's equations, see Von Laue, 1921) Here, however, we need only consider the terms linear in β and may neglect all higher-order terms in the Lorentz transformation. The kinematical equations of the Lorentz transformation reduce to the simple form

$$\underset{\sim}{r}' = \underset{\sim}{r} - \underset{\sim}{v}_o t, \quad t' = t, \quad \nabla' = \nabla, \quad \partial/\partial t' = \partial/\partial + v_o \cdot \nabla \qquad (3.1)$$

where $\underset{\sim}{v}_o$ is the velocity of the primed system with respect to the unprimed one and r the radius vector from the origin.

In the same approximation the field vectors transform as

$$\underset{\sim}{E}' = \underset{\sim}{E} + \underset{\sim}{v}_o \times \underset{\sim}{B}, \quad \underset{\sim}{B}' = \underset{\sim}{B} - \underset{\sim}{v}_o \times \underset{\sim}{E}/c^2$$

Introducing here the assumption that R_m is large which may be expressed by (2.10) we have

$$\left\{ v_o E/c^2 \right\} = \left\{ Bvv_o/c^2 \right\} = \left\{ B\beta^2 \right\}$$

provided $\underset{\sim}{v}_o$ and $\underset{\sim}{v}$ are of comparable order. Hence the transformation reduces to

$$\underset{\sim}{E}' = \underset{\sim}{E} + \underset{\sim}{v}_o \times \underset{\sim}{B}\ , \quad \underset{\sim}{B}' = \underset{\sim}{B} \tag{3.2}$$

The current density transforms to $\underset{\sim}{J}' = \underset{\sim}{J} + \eta \underset{\sim}{v}_o$. Since

$$\left\{ \eta\, v_o/\sigma E \right\} = \left\{ \Upsilon \right\} \text{ and } \left\{ \sigma E/J \right\} = \left\{ R_m \right\}$$

by the preceding section, we see that $\eta \underset{\sim}{v}_o$ is small of order ΥR_m. Now Υ is so exceedingly small that this product is small for all reasonable values of R_m; hence

$$\underset{\sim}{J}' = \underset{\sim}{J}\ , \quad \eta' = \eta \tag{3.3}$$

the second equation following from general principles of relativity. Furthermore, the conductivity, σ , can be shown to be a Lorentz invariant from general thermodynamical considerations (Von Laue, 1921). It is at once seen now that all the equations containing $\underset{\sim}{B}$ alone will not change under the transformations considered, nor will the ponderomotive force (2.24).

We next inquire into the space-charge, η . We must have conservation of charge which, on using the full expression (2.3), may be written

$$-\frac{\partial \eta}{\partial t} = \nabla \cdot (\underset{\sim}{J} - \varepsilon \frac{\partial \underset{\sim}{E}}{\partial t}) = \frac{\sigma}{\varepsilon}\, \eta + \sigma \nabla \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) + \nabla \cdot (\eta\, \underset{\sim}{v})$$

The last term is small and may be neglected, and we are left with a differential equation for η,

$$\dot{\eta} + (\sigma/\varepsilon)\,\eta = \sigma f(t) \tag{3.4}$$

where

$$-f(t) = \nabla \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) = \underset{\sim}{v} \cdot \nabla \times \underset{\sim}{B} - \underset{\sim}{B} \cdot \nabla \times \underset{\sim}{v} \tag{3.5}$$

does not in general vanish. The integral of (3.4) is

$$\eta(t) = \exp(-\sigma t/\varepsilon) \int_0^t dt\, f(t)\exp(\sigma t/\varepsilon)$$

Now the rate of change of f is determined by the frequencies, ω, of the fluid motion, and $\sigma/\varepsilon = \gamma\omega << \omega$. On letting therefore $f(t) = f(o) + tf'(o)$ the solution becomes, to within terms of the order of γ,

$$\eta(t) = \varepsilon f(o) + \varepsilon tf'(o)$$

Its meaning is apparent: the space charge is

$$\eta = -\varepsilon \nabla \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) \tag{3.6}$$

and as the parenthesis on the right changes with time, η follows this change quasistatically to within terms of the order of γ, that is synchronously to all practical purposes.

While thus hydromagnetic induction does in general produce a space charge in the conductor, (2.1) shows that the associated, longitudinal, part of the field does not affect the rate of change of $\underset{\sim}{B}$. With a proper choice of the constants of integration we can say that the longitudinal component of $\underset{\sim}{E}$, whose sources are η, is not accompanied by a magnetic field, a familiar result of conventional electrodynamics. Now we have seen before that the electrical energy density is small compared to the magnetic one, and similarly for the ponderomotive forces. For this reason we are not, in general, interested in the electric fields as such when dealing with problems of hydromagnetic theory. These fields become of physical interest only if perchance it can be shown that somewhere in the universe they give rise to the acceleration of elementary particles, a topic which is somewhat beyond the confines of our present subject. Thus it is altogether legitimate to ignore in the sequel the longitudinal component of $\underset{\sim}{E}$, even though it is of the same order of magnitude as the transverse component. If we

introduce a vector potential we may correspondingly set $\nabla \cdot \underset{\sim}{A} = 0$, as we have explained in Sec. 3, although this divergence is by no means small compared to $\nabla \times \underset{\sim}{A}$.

Cosmic matter is practically never an insulator in which static charges could be maintained over an appreciable length of time (the case of the formation of atmospheric thunderstorms by electrostatic effects being a notable exception); hence there will be no static charges other than those given by (3.6). It may be convenient to introduce static charges for mathematical reasons, e.g., at the boundary of a conductor against vacuum.

It is seen, then, that $\underset{\sim}{E}$ has both a divergence-free and an irrotational component; the former is given by the first equation (2.1), the latter can be expressed by (3.5) and the relation

$$\underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B} \tag{3.7}$$

which holds to within terms of the order of R_m^{-1}. Both components are as a rule of comparable order of magnitude. In dealing with problems of hydromagnetism, especially the dynamo theory, it is as a rule more convenient to work with equations that contain $\underset{\sim}{B}$ alone, so that questions concerning the electric field become irrelevant, especially since the ponderomotive force of the electric field or the electris stress tensors are by (2.7) negligibly small as compared with the corresponding magnetic quantities. In the hydromagnetic phenomena of rarefied gases the electric fields generated may become important for the acceleration of individual particles (cosmic rays), but these problems are not within the scope of the present review.

There is, however, one point that might be touched upon, namely the conception advanced by several authors, that the

conditions for the acceleration of particles from thermal, or in any event small, energies to higher energies are particularly favorable at so-called "neutral points". These points are defined so that $\underset{\sim}{B} = 0$ in the local frame of reference, that is, in a system of reference in which $\underset{\sim}{v} = 0$. It is correctly argued that in such a system the particles move in straight lines and that hence the conditions for setting up an electrical discharge are much more favorable than elsewhere where a discharge is quenched by the magnetic field (spiralling of the particles which effectively reduces their mean free path along $\underset{\sim}{E}$). Now if E_o designates the mean field at a neutral point and E the mean field in the fluid, we see from (2.10) that E_o is small, specifically

$$\left\{E_o\right\} = \left\{E/R_m\right\} = \left\{vB/R_m\right\} \qquad (3.8)$$

Since under most astrophysical conditions R_m is quite large, the theory of discharges at neutral points (Dungey, 1953) which at first sight is very attractive, needs a thorough revision in the light of this last result.

Previous to the development of hydromagnetic theory with its amplificatory mechanisms, the possibility of molecular electromotive forces as a cause of the large-scale electromagnetic fields has often been investigated. Thermoelectric potentials, potentials caused by pressure differences along a material boundary, and potentials caused by differential diffusion of negative and positive ions in a density gradient are typical of the mechanisms invoked. These theories (the writer pleads guilty to having once advanced one) have all had this in common: they use Ohm's law, $\underset{\sim}{J} = \sigma \underset{\sim}{E}$, in place of (2.8) to compute the currents. Now if we replace the first equation (2.1) by

$$\partial \underset{\sim}{B}/\partial t = -\nabla \times (\underset{\sim}{E} + \underset{\sim}{E}_i)$$

where $\underset{\sim}{E}_i$ is the impressed emf, our equation (2.11) becomes

$$\partial \underset{\sim}{B}/\partial t = -\nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) + \nu_m \nabla^2 \underset{\sim}{B} - \nabla \times \underset{\sim}{E}_i \tag{3.9}$$

In a turbulent medium the molecular magnetic diffusivity must be replaced by the eddy diffusivity which is very much larger; roughly,

$$\left\{ \nu_m' \right\} = \left\{ R_m \nu_m \right\} \tag{3.10}$$

By (2.12) this means that the conductivity is increased, or the resistivity decreased. This reduces the magnitude of the fields generated, both by the first and by the last term on the right of (3.9). Since, however, the existence of turbulence requires that there are large-scale motions on which the turbulence feeds, the first term on the right gives rise to large-scale hydromagnetic induction which is reduced but not in general wiped out by the turbulence, as we shall see in detail later. The effect of impressed emf's is different: the stationary current and the stationary magnetic field are established more rapidly when ν_m is larger, but the field produced by the saturation current is reduced in magnitude. From the last two terms of (3.9) the field corresponding to the stationary state is of the order

$$\left\{ B_i \right\} = \left\{ \lambda E_i / \nu_m \right\} \tag{3.11}$$

and according to (3.10) this field is smaller by a factor R_m under turbulent conditions. The theories referred to above have had great difficulties in accounting for an $\underset{\sim}{E}_i$ of sufficient magnitude to explain the observed fields, and these difficulties are multiplied if the turbulent resistivity is introduced. Most magnetic fields of large dimensions must be explained by dynamo action,

that is by hydromagnetic processes in which the field has been regenerated many times over. There is empirical evidence for this in the fact that almost all, if not all, stellar magnetic fields are time-dependent (Babcock and Cowling, 1953). Theoretically, we must say that the long decay periods for magnetic fields in stars, comparable to the age of the universe, which are computed in the basis of the molecular ν_m (Cowling, 1945) must be reduced by a factor $\{R_m\}$ which, in stars, is likely to amount to a high power of ten. Clearly, for a dynamo theory the way in which the magnetic field originated at the occasion of the first amplificatory processes becomes rather irrelevant as compared to the mechanism by which fields can be regenerated and maintained.

4. Free Aperiodic Modes.

The integration of the basic equation (2.11) is difficult even with a relatively simple geometry. We shall proceed to integrate two special cases: the case $\underset{\sim}{v} = 0$, representing free decay of a current system, essentially in a solid conductor, and the case $\nu_m = 0$ representing pure hydromagnetic induction. In this section we treat free decay; from (2.11) and (2.12) we now specialize to

$$\nabla^2 \underset{\sim}{B} - \mu\sigma \partial \underset{\sim}{B}/\partial t = 0 \tag{4.1}$$

To solve this equation we use the method of normal modes adapted to the aperiodic case (Elsasser, 1946/7) which differs in a number of particulars from the case of electromagnetic oscillations (Stratton, 1941). We set

$$\underset{\sim}{B}(\underset{\sim}{r},t) = \underset{\sim}{B}(\underset{\sim}{r})\exp(-\Lambda t) \tag{4.2}$$

and

$$\Lambda = k^2/\mu\sigma = k^2\nu_m$$

so that (4.1) becomes

$$\nabla^2 \underset{\sim}{B} + k^2 \underset{\sim}{B} = 0 \tag{4.4}$$

both Λ and k being assumed real. We have of course also

$$\nabla^2 \underset{\sim}{E} + k^2 \underset{\sim}{E} = 0 \tag{4.5}$$

From (4.3) the time of free decay is of the order

$$\left\{\Lambda^{-1}\right\} = \left\{\lambda^2/\nu_m\right\} \tag{4.6}$$

increasing with the square of the linear dimensions. ν_m is of the general order of unity (mks) for metallic conductors, permitting a ready estimate of the order of the decay times. These

times are fictitious, as we have pointed out before, since actually we must replace ν_m by the magnetic eddy viscosity.

The conventional scalar wave equation can be solved by separation of variables in 11 different systems of coordinates (Stratton, 1941). From these solutions one obtains sets of orthogonal modes by imposing suitable boundary conditions. For reasons too complex to be stated briefly the method cannot readily be extended to the vectorial wave equation; both the separation of variables and the boundary conditions present special mathematical difficulties. Systems of modes can be found for three types of geometry: plane waves, cylindrical waves and spherical waves. The theory has grown up in a more or less ad hoc fashion, though Stratton attempts some systematization, following earlier work by Mie, Debye and Hansen. We have tried to develop the method used to solve the vector wave equation and the formalism required for boundary conditions and orthogonality in a fairly general fashion, so as to exhibit both the scope and the limitations of the method. The ensuing formulas may be used in all the known cases of solutions. We then proceed to construct, for spherical boundary conditions, the actual solutions and to discuss in detail orthogonality and normalization.

Vector Wave Formalism.

In order to construct solutions of the vector wave equation we start from the scalar wave equation

$$\nabla^2\psi + k^2\psi = 0 \tag{4.7}$$

and shall derive from every solution of this equation three associated solutions of the vector wave equation (4.4); one of

them is of the form

$$\underset{\sim}{U} = \nabla\psi \,, \quad \nabla\nabla\cdot\underset{\sim}{U} + k^2\underset{\sim}{U} = 0 \tag{4.8}$$

where ψ is any suitable solution of (4.7); these are the longitudinal waves of vanishing curl. A second solution is of the form

$$\underset{\sim}{T} = \nabla \times \psi\underset{\sim}{Z} \tag{4.9}$$

where $\underset{\sim}{Z}$ is a vector field to be specified later. A third solution, finally, is

$$k\underset{\sim}{S} = \nabla \times \underset{\sim}{T} \tag{4.10}$$

The vector fields (4.9) and (4.10) are transverse, of vanishing divergence. There seems to be no simple way other than (4.9) and (4.10) to derive a pair of linearly independent transverse vectors from a scalar.

We can rewrite (4.9) as

$$T = \nabla\psi \times \underset{\sim}{Z} + \psi\nabla \times \underset{\sim}{Z} \tag{4.11}$$

On substituting this into (4.10) we find after some straightforward calculations, using (4.7),

$$\begin{aligned} k\underset{\sim}{S} = {} & k^2\psi\underset{\sim}{Z} + \nabla(\nabla\psi\cdot\underset{\sim}{Z}) \\ & + \nabla\psi(\nabla\cdot\underset{\sim}{Z}) - 2(\nabla\psi\cdot\nabla)\underset{\sim}{Z} + \psi\nabla\times\nabla\times\underset{\sim}{Z} \end{aligned} \tag{4.12}$$

Again, since $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are assumed to obey the wave equation (4.4) which may be written

$$\nabla\times\nabla\times\underset{\sim}{S} - k^2\underset{\sim}{S} = 0 \,, \quad \nabla\times\nabla\times\underset{\sim}{T} - k^2\underset{\sim}{T} = 0 \tag{4.13}$$

we have from (4.10) and the second of (4.13)

$$\nabla \times \underset{\sim}{S} = k\underset{\sim}{T} \tag{4.14}$$

Equations (4.10) and (4.14) exhibit a characteristic symmetry between the vectors $\underset{\sim}{S}$ and $\underset{\sim}{T}$. Next, from (4.9) and (4.14)

$$\nabla \times \underset{\sim}{S} = k \nabla \times \psi \underset{\sim}{Z} \tag{4.15}$$

On the other hand, the curl of (4.12) agrees with (4.15) only if the last three terms in (4.12) are the gradient of a scalar. We shall consider $\underset{\sim}{Z}$ as a quantity determined by the particular boundary conditions, hence related to the system of coordinates chosen, whereas ψ is any one of a set of orthogonal modes. For the present purpose we may therefore assume ψ to be effectively arbitrary, and $\underset{\sim}{Z}$ and ψ as independent of each other. It follows that the last term in (4.12) must be the gradient of a scalar in itself since it contains ψ, whereas the two other terms contain the derivatives of ψ. These two terms both involve $\nabla\psi$; they have arbitrary but in general different directions; hence each of them must again be the gradient of a scalar separately. From the first and the third of the three terms there follows $\nabla \cdot \underset{\sim}{Z}$ = const., and $\nabla \times \underset{\sim}{Z}$ = const., respectively; hence $\underset{\sim}{Z}$ is a linear function of the cartesian coordinates. Combining this with the requirement that the middle term, $(\nabla\psi \cdot \nabla)\underset{\sim}{Z}$ must be a gradient we find two solutions: One is clearly $\underset{\sim}{Z}$ = const.; we may for instance take $\underset{\sim}{Z}$ to be a unit vector in the z-direction. The other solution is $\underset{\sim}{Z} = \underset{\sim}{r}$, where $\underset{\sim}{r}$ is the radius vector from the origin. This latter choice gives $(\nabla\psi \cdot \nabla)\underset{\sim}{r} = \nabla\psi$. The former choice is used with plane and cylindrical waves, the cylinder axis being in the z-direction; the latter is appropriate for spherical waves. We then have from the preceding formulas, for constant $\underset{\sim}{Z}$

$$\underset{\sim}{T} = \nabla\psi \times \underset{\sim}{Z}\,; \quad \underset{\sim}{S} = k\psi\underset{\sim}{Z} + k^{-1}\nabla(\partial\psi/\partial z) \tag{4.16}$$

and for $\underset{\sim}{Z} = \underset{\sim}{r}$, in polar coordinates

$$\underset{\sim}{T} = \nabla\psi \times \underset{\sim}{r}\,, \quad \underset{\sim}{S} = k\psi\underset{\sim}{r} + k^{-1}\nabla(\partial r\psi/\partial r) \tag{4.17}$$

We may summarize the last two formulas by

$$\underset{\sim}{T} = \nabla\psi \times \underset{\sim}{Z}, \quad \underset{\sim}{S} = k\psi\underset{\sim}{Z} + k^{-1}\nabla(\psi)' \tag{4.18}$$

We shall now write down formulas from which the orthogonality relations of these vector modes may be derived for appropriate boundary conditions. Individual scalar modes which satisfy (4.5) will be distinguished by indices, $\psi_1\psi_2 \ldots$; the vector modes derived from ψ_1 by (4.18) and (4.8) will be designated by S_1, T_1, U_1. We now have

$$\begin{aligned}\int \underset{\sim}{U}_1 \cdot \underset{\sim}{T}_2 dV &= \int \nabla\psi_1 \times \nabla\psi_2 \cdot \underset{\sim}{Z}\, dV = \int \nabla \times (\psi_1\nabla\psi_2) \cdot \underset{\sim}{Z}\, dV \\ &= \int \nabla \cdot \left[\psi_1\nabla\psi_2 \times \underset{\sim}{Z}\right] dV = \int \psi_1\nabla\psi_2 \times \underset{\sim}{Z} \cdot \underset{\sim}{n}\, d\sigma\end{aligned} \tag{4.19}$$

where $\underset{\sim}{n}$ is the unit normal to the surface, directed outwards. Next we find, using Green's theorem

$$\begin{aligned}\int \underset{\sim}{U}_1 \cdot \underset{\sim}{S}_2 dV &= k_2\int \psi_2\nabla\psi_1 \cdot \underset{\sim}{Z}\, dV + k_2^{-1}\int \nabla\psi_1 \cdot \nabla(\psi_2)' dV \\ &= k_2\int \psi_2\nabla\psi_1 \cdot \underset{\sim}{Z}\, dV + k_2^{-1}k_1^{2}\int \psi_1\psi_2 dV + k_2^{-1}\int (\psi_2)'(\partial\psi_1/\partial n)d\sigma\end{aligned} \tag{4.20}$$

and finally

$$\begin{aligned}\int \underset{\sim}{T}_1 \cdot \underset{\sim}{S}_2 dV &= k_2^{-1}\int \nabla(\psi_2)' \times \nabla\psi_1 \cdot \underset{\sim}{Z}\, dV \\ &= k_2^{-1}\int \nabla \cdot \left[(\psi_2)'\nabla\psi_1 \times \underset{\sim}{Z}\right] dV = k_2^{-1}\int (\psi_2)'\nabla\psi_1 \times \underset{\sim}{Z} \cdot \underset{\sim}{n}\, d\sigma\end{aligned} \tag{4.21}$$

Some further formulas are required for the orthogonality relations among vectors of the same type. For the $\underset{\sim}{U}$-vectors we can transform the integral over $\nabla\psi_1 \cdot \nabla\psi_2$ by means of the conventional Green's theorem as in (4.20). For the $\underset{\sim}{T}$-vectors we find from (4.18)

$$\int \underset{\sim}{T}_1 \cdot \underset{\sim}{T}_2 dV = \int (\nabla\psi_1 \cdot \nabla\psi_2)\underset{\sim}{Z}^2 dV - \int (\underset{\sim}{Z} \cdot \nabla\psi_1)(\underset{\sim}{Z} \cdot \nabla\psi_2) dV \tag{4.22}$$

Further useful relations follow from the vectorial equivalent of Green's formulas (Stratton, 1941). For two arbitrary vectors, provided $\nabla \cdot \mathbf{B} = 0$,

$$\int\left[(\nabla \times \mathbf{A})\cdot(\nabla \times \mathbf{B}) + \mathbf{A}\nabla^2\mathbf{B}\right]dV = \int \mathbf{A} \times (\nabla \times \mathbf{B}) \cdot \mathbf{n}\,d\sigma \qquad (4.23)$$

which may readily be proved on converting the right-hand side into a volume integral by Gauss' theorem. The equivalent of Green's second formula is

$$\int\left[\mathbf{A}\cdot\nabla^2\mathbf{B}-\mathbf{B}\cdot\nabla^2\mathbf{A}\right]dV = \int\left[\mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A})\right]\cdot dn\sigma \qquad (4.24)$$

provided $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{B} = 0$. From (4.23) we obtain an expression relating the orthogonality of the $\mathbf{S}$-vectors to that of the $\mathbf{T}$-vectors:

$$k_1 k_2 \int \mathbf{S}_1 \cdot \mathbf{S}_2 dV = k_2^2 \int \mathbf{T}_1 \cdot \mathbf{T}_2 dV + k_2 \int \mathbf{S}_1 \times \mathbf{T}_2 \cdot \mathbf{n}\,d\sigma \qquad (4.25)$$

The surface integral on the right can be further reduced by means of the identity, following from (4.18)

$$\int \mathbf{S}_1 \times \mathbf{T}_2 \cdot \mathbf{n}\,d\sigma = \int \mathbf{T}_2 \cdot \mathbf{n} \times \mathbf{S}_1\,d\sigma$$

$$= k_1 \int \psi_1(\nabla\psi_2 \times \mathbf{Z})\cdot(\mathbf{n} \times \mathbf{Z})d\sigma - k_1^{-1}\int(\nabla\psi_2 \times \mathbf{Z})\cdot(\nabla(\psi_1)' \times \mathbf{n})d\sigma \qquad (4.26)$$

From (4.24) we get the two useful relations

$$(k_1^2 - k_2^2)\int \mathbf{T}_1 \cdot \mathbf{T}_2 dV = \int\left[k_2 \mathbf{T}_1 \times \mathbf{S}_2 - k_1 \mathbf{T}_2 \times \mathbf{S}_1\right] \cdot \mathbf{n}\,d\sigma \qquad (4.27)$$

$$(k_1^2 - k_2^2)\int \mathbf{S}_1 \cdot \mathbf{S}_2 dV = \int\left[k_2 \mathbf{S}_1 \times \mathbf{T}_2 - k_1 \mathbf{S}_2 \times \mathbf{T}_1\right] \cdot \mathbf{n}\,d\sigma \qquad (4.28)$$

If our conducting body is of finite size (rectangular box or cylindric prism for cartesian coordinates, circular or elliptic cylinder for cylindrical coordinates, sphere for polar coordinates) we must join our solutions for the inside to the solutions of Maxwell's equations for free space at the outside. Assuming that ε and μ have the same values inside the conductor

as outside, the boundary conditions impose continuity of all field vectors, except that there may be a surface charge, τ, per unit area so that for the normal components

$$(E_n)_{outside} - (E_n)_{inside} = \tau/\varepsilon \qquad (4.29)$$

In outer space we have a field which adjusts itself quasistatically to the fields that prevail at the boundary and that are the result of the validity of (4.1) and (2.1) on the inside. Since the displacement current is negligible, we have from (2.1) and (2.2) for the outside field

$$\nabla^2 \underset{\sim}{B} = 0\ ,\quad \nabla^2 \underset{\sim}{E} = 0\ ,\quad \nabla \cdot \underset{\sim}{B} = 0\ ,\quad \nabla \cdot \underset{\sim}{E} = 0 \qquad (4.30)$$

It is not, in general, possible to choose boundary conditions for the solutions of the vector wave equation such that a fully orthogonal system of vectors $\underset{\sim}{S}$, $\underset{\sim}{T}$, $\underset{\sim}{U}$ results. The boundary conditions on ψ are of course determined by the boundary conditions of the electromagnetic field quantities; it then appears that "almost" all the modes are mutually orthogonal, but that for certain pairs of vectors the orthogonality fails (Stratton, 1941). Fortunately, this failure is sufficiently limited so that it does not seriously hamper the use of the formalism for the solution of our physical problems. There is one case where full orthogonality obviously obtains: for plane waves in a rectangular box with cyclic boundary conditions at all faces.

Modes of the Sphere.

We shall be principally interested in spherical conductors. We assume uniform conductivity and let R be the radius of the sphere. We shall assume the outer space to be vacuum. Consider

first the inside. We have for the solutions of (4.7), apart from an arbitrary constant,

$$\psi = j_n(k_{ns}r)Y_n^m(\vartheta, \varphi) \tag{4.31}$$

where j_n is a spherical Bessel function,

$$j_n(x) = (\pi/2x)^{1/2}J_{n+1/2}(x) \tag{4.32}$$

and where the k_{ns} are determined by the boundary conditions. The Y_n^m are the conventional surface harmoncis. We now write down the expressions for the vector modes in polar coordinates in terms of ψ. They are given by (4.8) and by (4.17) which yield

$$\left.\begin{aligned} \underset{\sim}{U}_r = \partial\psi/\partial r \;, \quad \underset{\sim}{U}_\vartheta = r^{-1}\partial\psi/\partial\vartheta \\ \underset{\sim}{U}_\varphi = (r \sin\vartheta)^{-1}\partial\psi/\partial\varphi \end{aligned}\right\} \tag{4.33}$$

This type of mode is purely longitudinal. Next we have

$$\left.\begin{aligned} \underset{\sim}{T}_r = 0 \;, \quad \underset{\sim}{T}_\vartheta = (\sin\vartheta)^{-1}\partial\psi/\partial\varphi \\ \underset{\sim}{T}_\varphi = -\partial\psi/\partial\vartheta \end{aligned}\right\} \tag{4.34}$$

This type of mode will be designated as toroidal. Finally

$$\left.\begin{aligned} \underset{\sim}{S}_r &= kr\psi + k^{-1}\partial^2(r\psi)/\partial r^2 = n(n+1)(kr)^{-1}\psi \\ \underset{\sim}{S} &= (kr)^{-1}\partial^2(r\psi)/\partial r\partial\vartheta \\ \underset{\sim}{S}_\varphi &= (kr \sin\vartheta)^{-1}\partial^2(r\psi)/\partial r\partial\varphi \end{aligned}\right\} \tag{4.35}$$

This type of mode will be designated as poloidal.

For the outer space the field equations reduce to the vectorial Laplace equations (4.30); the corresponding generating scalar is

$$\psi = C\, r^{-n-1}Y_n^m(\vartheta, \varphi) \tag{4.36}$$

where C will be chosen so as to assure continuity with the inside solution by virtue of the electromagnetic boundary conditions.

The solution $\underset{\sim}{U}$ defined by (4.8) or (4.33) continues to hold in the limit k = 0. In the case of the transverse modes, however, we must redefine our expressions so that (4.35) remains finite for k = 0. This is easily done by considering the vectors $k\underset{\sim}{S}$ and $k\underset{\sim}{T}$ in place of the above. Then $k\underset{\sim}{T}$ vanishes as k goes to zero and one readily verifies that $k\underset{\sim}{S} = -n\underset{\sim}{U}$; this, furthermore, agrees with (4.14), namely $\underset{\sim}{T} = \nabla \times (\underset{\sim}{S}/k) = 0$. Hence on the outside there exists only one solution which is essentially longitudinal, but may if desired be expressed in terms of a poloidal vector field with the generating function (4.36).

We shall now construct the free, aperiodically damped electromagnetic modes. Again, $\underset{\sim}{B}$, $\underset{\sim}{E}$ will designate field vectors from which the time factor has been split off, by (4.2). The longitudinal modes (4.33) are readily constructed, but they are purely electrostatic and the corresponding magnetic field vanishes. As indicated in Sec. 3 they are only of subordinate interest for the dynamo problem; their formalism is moreover quite straightforward.

For the transverse modes we have from the field equations together with (4.2) and (4.3)

$$\nabla \times \underset{\sim}{B} = k(\mu\sigma\underset{\sim}{E}/k); \quad \nabla \times (\mu\sigma\underset{\sim}{E}/k) = k\underset{\sim}{B} \tag{4.37}$$

If we introduce a vector potential it is related to $\underset{\sim}{E}$ by

$$\underset{\sim}{A} = \underset{\sim}{E}/\Lambda = \mu\sigma\underset{\sim}{E}/k^2$$

The equations (4.37) are identical in form with (4.10) and (4.14). Moreover, the vector wave equation (4.13) is a

consequence of these relations. There are two types of transverse modes: the toroidal magnetic modes where $\underset{\sim}{B}$ is of type $\underset{\sim}{T}$, and the poloidal magnetic modes where $\underset{\sim}{B}$ is of type $\underset{\sim}{S}$.

We first consider the toroidal magnetic modes. To obtain an idea of their geometry we note from (4.34) that for rotational symmetry (zonal harmonics) there is only a φ-component; hence the magnetic lines of force coincide with the circles of latitude. For the toroidal dipole mode $\underset{\sim}{B}_\varphi$ has the same sign throughout, whereas for the toroidal quadrupole mode $\underset{\sim}{B}_\varphi$ changes sign at the equator.

We have seen that the $\underset{\sim}{T}$-vectors vanish identically in outer space where $\sigma = 0$. Hence the boundary condition is $\underset{\sim}{B} = 0$ at the surface of the conducting sphere, $r = R$. From (4.34) and (4.31) this gives the characteristic equation

$$j_n(k^t_{ns}R) = 0 \tag{4.39}$$

The k^t_{ns} form a twofold sequence, depending on the "quantum numbers" n and s, the latter numbering the successive zeros of j_n. The electric field corresponding to this mode is poloidal; by (4.35) its normal component vanishes at $r = R$, but the tangential components do not in general vanish. We can fulfill the boundary conditions by combining an external multipole field with a surface charge defined by (4.29). (It should be pointed out that the transverse vector modes start with $n = 1$; the solutions corresponding to $n = 0$ vanish.)

We next consider the poloidal magnetic modes. For zonal harmonics, that is rotational symmetry, the magnetic field lines are confined to the meridional planes. To fulfill the boundary

condition for $\underset{\sim}{B}$ for the general mode of this type we must have continuity of all three components at the surface of the conductor. It is convenient to express the external multipole field in terms of the poloidal vectors (4.35) rather than by the longitudinal $\underset{\sim}{U}$-vectors; we have shown above that this is legitimate. Now for this external field whose generating scalar is (4.36) the relation of the tangential to the normal components of $\underset{\sim}{S}$ is given by $\partial(r\psi)/\partial r = -n\psi$. If there is to be continuity of all components, the internal ψ must obey the same condition at the surface, $r = R$, that is

$$\left[\frac{\partial(rj_n)}{\partial r} + nj_n\right]_{r = R} = 0 \tag{4.40}$$

This is the characteristic equation for these modes which may readily be transformed into

$$j_{n-1}(k^p_{ns}R) = 0 \tag{4.41}$$

Moreover, if (4.36) is the generating scalar for the external $\underset{\sim}{S}$-vectors we obtain

$$c_{ns} = R^{n+1}j_n(k_{ns}R) \tag{4.42}$$

The electric field of these modes is purely toroidal. Since there can be no toroidal field in empty outer space, the boundary conditions for $\underset{\sim}{E}$ cannot be fulfilled; in order to satisfy them we would have to go to a higher-order approximation. Since the aperiodic modes decay extremely slowly, we could try to fulfill the boundary conditions by assuming a minute electrical conductivity in outer space which would make a toroidal electric field possible and would also correspond fairly closely to geophysical and astrophysical conditions. In any event, the electric field

is again negligibly small.

We next discuss the orthogonality and normalization of the modes. From (4.20) and (4.21) one readily verifies that all $\underset{\sim}{T}$-vectors are orthogonal to all $\underset{\sim}{S}$-vectors over the interior of the sphere. Similarly, it follows from (4.19) that all $\underset{\sim}{T}$-vectors are orthogonal to all $\underset{\sim}{U}$-vectors. The $\underset{\sim}{S}$-vectors are, however, not orthogonal to the $\underset{\sim}{U}$-vectors (Stratton, 1941) but since we are not interested in the longitudinal field components this fact will not impede our calculations. Next, consider the mutual orthogonality and normalization of the toroidal modes. We introduce the abbreviation ∇' by

$$\nabla = r^{-1}\underset{\sim}{r}(\partial/\partial r) + r^{-1}\nabla' \qquad (4.43)$$

so that ∇' is a gradient vector along the surface of the unit sphere. Applying (4.43) and (4.31) to (4.22) we find

$$\int \underset{\sim}{T}_1 \cdot \underset{\sim}{T}_2 dV = \int j_1 j_2 r^2 dr \int \nabla' Y_1 \cdot \nabla' Y_2 d\sigma \qquad (4.44)$$

To evaluate the surface integral, consider for a moment a potential function, $\psi = r^n Y_n^m$. Applying Green's theorem to two such functions we find, on using (4.43),

$$\int r^{-2}\nabla'\psi_1 \cdot \nabla'\psi_2 dV + \int \frac{\partial\psi_1}{\partial r}\frac{\partial\psi_2}{\partial r} dV = \int \psi_1 \frac{\partial\psi_2}{\partial n} d\sigma$$

If we now choose as the volume of integration the interior of the unit sphere, the integrals are readily evaluated and yield, provided we use complex harmonics, $Y_n^m = P_n^m(\cos\vartheta)e^{im\varphi}$,

$$X = \int \nabla' Y_1 \cdot \nabla' Y_2 d\sigma = 0 \text{ for } Y_1 \neq Y_2^* \qquad (4.45)$$

where the asterisk designates the conjugate complex, and

$$X = \int |\nabla' Y_n^m|^2 d\sigma = n(n+1) \int |Y_n^m|^2 d\sigma$$
$$= \frac{4\pi n(n+1)}{2n+1} \cdot \frac{(n+m)!}{(n-m)!} \qquad (4.46)$$

In order to prove orthogonality of the radial functions for fixed n and variable x we require the boundary condition for the toroidal modes which is $j_n = 0$ at the surface of the sphere, by (4.39). With this, the radial integral of (4.44) may be evaluated from the formula, proved in the theory of Bessel functions (see Jahnke-Emde, Tables of Functions)

$$\int j_n^2(kx)x^2dx = (x^3/2)\left[j_n^2(kx) - j_{n-1}(kx)j_{n+1}(kx)\right] \qquad (4.47)$$

which now reduces to

$$\int_0^R \left[j_n(k_{ns}^2 r)\right]^2 r^2 dr = -(R^3/2)j_{n-1}(k_{ns}^t R)j_{n+1}(k_{ns}^t R) \qquad (4.48)$$

The right-hand side is of course essentially positive in spite of the minus sign in front.

For the poloidal modes we again discuss first orthogonality with respect to the spherical harmonics. Using (4.43) in (4.17) we see that $\underset{\sim}{S} = a\underset{\sim}{r}Y + b\nabla'Y$ where a and b are functions of r, and orthogonality with respect to the Y's follows from (4.45). To establish the orthogonality of the radial functions and the normalization, we first derive from (4.28), (4.26) and (4.43) the relation

$$\int \underset{\sim}{S}_1\cdot\underset{\sim}{S}_2 dV = \frac{X}{R(k_1^2-k_2^2)}\left[\frac{k_1}{k_2}j_1\,\frac{\partial(rj_2)}{\partial r} - \frac{k_2}{k_1}\,j_2\,\frac{\partial(rj_1)}{\partial r}\right]_{r\,=\,R} \qquad (4.49)$$

where X is the surface integral (4.45-46). If we multiply the bracket by k_1k_2 and substitute the boundary condition (4.40) the right-hand side is seen to vanish for $n_1 = n_2$, provided however, $k_1 \neq k_2$; for $k_1 = k_2$ the formula becomes invalid. Thus the poloidal modes are orthogonal over the <u>interior</u> of the sphere. Since for given n the different radial modes have the same external field apart from constant factors, the modes are not orthogonal on

integration over all space. For $k_1 = k_2$ we use (4.25) and (4.26) and obtain

$$\int |S|^2 dV = \int |T|^2 dV - \frac{1}{k^2}\left[r j_n \frac{\partial(r j_n)}{\partial r}\right]_{r=R} \int |\nabla' Y|^2 d\sigma$$

The first integral on the right is given by (4.44) but to evaluate it by (4.47) we must now use the boundary condition (4.41). At the same time we can simplify the bracket on the right by means of the form (4.40) of this boundary condition. The result is

$$\int_{\text{inside}} |\underline{S}|^2 dV = \left[R^3/2 + nR/k_{ns}^2\right] j_n^2(k_{ns}R) \int |\nabla' Y|^2 d\sigma \qquad (4.50)$$

We finally compute the overlap of the external fields for the same n but different values of s. Since most of the preceding general formulas break down if applied to the outer space, the calculations are conveniently carried out directly from (4.35) on using the generating scalar defined by (4.36) and (4.42). The result is

$$\int_{\text{outside}} |\underline{S}|^2 dV = (nR/k_1 k_2) j_1(k_1R) j_2(k_2R) \int |\nabla' Y|^2 d\sigma \qquad (4.51)$$

For $k_1 = k_2$ this corresponds exactly to the second term in the bracket of (4.50). These expressions represent of course twice the magnetic energy of such a mode on the inside and the outside, respectively. Hence the ratio of the outside to the inside energy is

$$E_{\text{outside}}/E_{\text{inside}} = \left[1 + (k_{ns}R)^2/2n\right]^{-1} \qquad (4.52)$$

Let us finally obtain an estimate of the decay time of these modes for the earth's metallic core. We shall see later that the toroidal dipole modes cannot be appreciably excited, the most significant modes being the poloidal dipoles and the toroidal quadrupoles. For the former we have $j_o(kR) = 0$ and for the latter $j_2(kR) = 0$. The lowest root for the dipole is $kR = \pi$, the lowest root for the toroidal quadrupole is $kR = 5.8$. From (4.2)

and (4.3) the decay time is $\mu\sigma/k^2$. With $R = 3.5 \cdot 10^6$ meters and $\sigma = 10^6$ mks, one tenth of the conductivity of ordinary iron, we obtain 50,000 years and 14,000 years respectively. These figures are of course purely nominal, since the actual decay times are determined by the magnetic eddy diffusivity and are no doubt much smaller, perhaps closer in order of magnitude to 1000 years, as may be judged from certain features of the geomagnetic secular variation (Elsasser, 1950).

5. Kinematics of Induction.

We now consider the induction equation in the absence of dissipative losses; by (2.11)

$$\partial \underset{\sim}{B}/\partial t = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \qquad (5.1)$$

or its equivalent (2.15), namely,

$$d(\rho^{-1}\underset{\sim}{B})/dt = (\rho^{-1}\underset{\sim}{B} \cdot \nabla)\underset{\sim}{v} \qquad (5.2)$$

and the corresponding integral theorem (2.14) which is

$$d/dt \int B_n d\sigma = 0 \qquad (5.3)$$

In terms of the vector potential we have from (2.28)

$$\partial \underset{\sim}{A}/\partial t = \underset{\sim}{v} \times (\nabla \times \underset{\sim}{A}) \qquad (5.4)$$

which may be written alternately

$$d\underset{\sim}{A}/dt = \underset{\sim}{v} \times (\nabla \times \underset{\sim}{A}) + (\underset{\sim}{v} \cdot \nabla)\underset{\sim}{A} \qquad (5.5)$$

and (5.3) becomes

$$d/dt \int \underset{\sim}{A} \cdot d\underset{\sim}{C} = 0 \qquad (5.6)$$

It is at once apparent that the induction process does not involve any material constants of the medium. The relative rate of change of the magnetic field is of the order $\{ v/\lambda \}$; thus if we wait for a time during which a fluid particle travels a distance of the order of the linear dimensions of the system, the amplification of the field can become appreciable; for much longer times the amplification may become very large under conditions otherwise suitable which will be discussed later.

It has been found by Parker (1954) that the conservation equations given above can be integrated with respect to the time. In order to avoid some of the mathematical complexities of the full tensor-analytical treatment we shall adopt a mixed formalism in which we pass from vector to tensor notation as required.

We integrate (5.2) over a volume, getting

$$\int \rho(d/dt)(\underset{\sim}{B}/\rho)dV = \int (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{v}\, dV \qquad (5.7)$$

Now we may show that

$$\int (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{v}\, dV = \int \underset{\sim}{v}\, B_n d\sigma \qquad (5.8)$$

To prove this, consider the x-component; we have the identity

$$\underset{\sim}{B} \cdot \nabla v_x = \nabla \cdot (v_x \underset{\sim}{B}) - v_x(\nabla \cdot \underset{\sim}{B})$$

and the last term vanishes. On applying Gauss' theorem on the right, (5.8) follows.

Let us assume that the volume of integration is attached to the fluid particles and moves with the latter. Then $B_n d\sigma$ is an invariant by (5.3) and may be written $(B_n d\sigma)^o$ where the superscript o will here and in the sequel designate the values of variables at an initial instant, t = 0. Thus we shall write $\underset{\sim}{B}^o$, $\underset{\sim}{v}^o$, $\underset{\sim}{r}^o$ for field, velocity, and position pertianing to a particle at time t = 0. This is essentially the Lagrangian method of hydrodynamics where the variables referring to time t are considered as functions of the variables which characterize the same fluid particle at time t = 0. Thus $\underset{\sim}{B}$, $\underset{\sim}{v}$, $\underset{\sim}{r}$ are functions of $\underset{\sim}{B}^o$, $\underset{\sim}{v}^o$, $\underset{\sim}{r}^o$ and of t; in components we consider B_i, v_i, x_i as functions of the initial values B_i^o, v_i^o, x_i^o and of the time. To simplify the formalism we shall confine ourselves for the present

to cartesian coordinates. We write ∇^o for the operator with the components $\partial/\partial x_i^o$.

Now since $B_n d\sigma = B_n^o d\sigma^o$ by (5.3), we see from (5.8) that

$$\int (\underset{\sim}{B} \cdot \nabla) v \, dV = \int (\underset{\sim}{B}^o \cdot \nabla^o) \underset{\sim}{v} \, dV^o$$

where on the right v is considered a function of the x_i^o. Since $\rho \, dV = \rho^o dV^o$ this gives

$$\int [(\underset{\sim}{B} \cdot \nabla) - (\rho/\rho^o)(\underset{\sim}{B}^o \cdot \nabla^o)] \underset{\sim}{v} \, dV = 0$$

Here we can equate the integrand to zero since the volume of integration is arbitrary. If we substitute in (5.7) we find

$$d/dt(\underset{\sim}{B}/\rho) = [(B^o/\rho^o) \cdot \nabla^o] \underset{\sim}{v} \tag{5.9}$$

This integrates to

$$\underset{\sim}{B}/\rho - B^o/\rho^o = [(\underset{\sim}{B}^o/\rho^o) \cdot \nabla^o](\underset{\sim}{r} - \underset{\sim}{r}^o)$$

But since $\underset{\sim}{B}^o = (\underset{\sim}{B}^o \cdot \nabla^o)\underset{\sim}{r}^o$, identiallly, this reduces to

$$\underset{\sim}{B}/\rho = [(\underset{\sim}{B}^o/\rho^o) \cdot \nabla^o]\underset{\sim}{r} \tag{5.10}$$

which is the desired integral, expressing $\underset{\sim}{B}$ at time t in terms of $\underset{\sim}{B}^o$ and the kinematical properties of the fluid.

We next discuss the conservation theorem for the vector potential.*) Equation (5.5) is closely related to a well-known vector-analytical identity. If we let

$$\nabla(\underset{\sim}{v} \cdot \underset{\sim}{A}) = [\nabla(\underset{\sim}{v} \cdot \underset{\sim}{A})]_{v=\text{const.}} + [\nabla(\underset{\sim}{v} \cdot \underset{\sim}{A})]_{A=\text{const.}} \tag{5.11}$$

We find readily that (5.5) may be written

$$d\underset{\sim}{A}/dt = [\nabla(\underset{\sim}{v} \cdot \underset{\sim}{A})]_{v=\text{const.}} \tag{5.12}$$

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* For several of the subsequent formulas of this section the author is indebted to Dr. Willaim L. Bade.

If we substitute (5.12) into (5.11) we get

$$dA/dt = -A \times (\nabla \times v) - (A \cdot \nabla)v + \nabla\psi \qquad (5.13)$$

where $\psi = v \cdot A$ may usually be ignored for the reasons explained in Sec. 3. These expressions may be greatly simplified on using tensor notation. By viture of (5.11) the relations (5.12) and (5.13) become

$$dA_i/dt - \sum_k v_k(\partial A_k/\partial x_i) = -\sum_k A_k(\partial v_k/\partial x_i) + \partial\psi/\partial x_i \qquad (5.14)$$

To obtain the time-integral of the conservation equation it is best to start over again from (5.6) which may be integrated directly giving, in tensor notation,

$$\sum_i \int (A_i dc_i - A_i^o dc_i^o) = 0 \qquad (5.15)$$

But

$$dc_i^o = \sum_k dc_k(\partial x_i^o/\partial x_k)$$

On substituting this into (5.15) the integrand must be the gradient of a scalar:

$$A_i = \sum_k A_k^o(\partial x_k^o/\partial x_i) + \partial\psi'/\partial x_i \qquad (5.16)$$

which is the desired integral for A: it may be compared with the corresponding formula (5.10) for B which, in the present notation, reads

$$B_i/\rho = \sum_k (B_k^o/\rho^o)(\partial x_i/\partial x_k^o) \qquad (5.17)$$

If we drop the gradient term in (5.16), we obtain a simple relation on forming the scalar product of A and B/ρ, namely,

$$\underset{\sim}{A} \cdot \underset{\sim}{B}/\rho = \underset{\sim}{A}^{o} \cdot \underset{\sim}{B}^{o}/\rho^{o}, \quad (d/dt)(\underset{\sim}{A} \cdot \underset{\sim}{B}/\rho) = 0 \tag{5.18}$$

So far we have confined ourselves to cartesian coordinates. For reference purposes we give the expression of the preceding integral theorems in cylindrical and in spherical polar, coordinates. We shall omit the density ρ; to reintroduce it, it is merely necessary to replace $\underset{\sim}{B}$ by $\underset{\sim}{B}/\rho$ and $\underset{\sim}{B}^{o}$ by $\underset{\sim}{B}^{o}/\rho^{o}$. The calculations are straightforward differential geometry and need not be described. In cylindrical coordinates, ρ, φ, z (where $\rho = (x^2 + y^2)^{1/2}$ is of course not to be confounded with the density) the field transforms as

$$\begin{aligned}
B_\rho &= \frac{\partial \rho}{\partial \rho^o} B^o_\rho + \frac{1}{\rho^o}\frac{\partial \rho}{\partial \varphi^o} B^o_\varphi + \frac{\partial \rho}{\partial z^o} B^o_z \\
B_\varphi &= \rho \frac{\partial \varphi}{\partial \rho^o} B^o_\rho + \frac{\rho}{\rho^o}\frac{\partial \varphi}{\partial \varphi^o} B^o_\varphi + \rho \frac{\partial \varphi}{\partial z^o} B^o_z \\
B_z &= \frac{\partial z}{\partial \rho^o} B^o_\rho + \frac{1}{\rho^o}\frac{\partial z}{\partial \varphi^o} B^o_\varphi + \frac{\partial z}{\partial z^o} B^o_z
\end{aligned} \tag{5.19}$$

and the vector potential as

$$\begin{aligned}
A_\rho &= \frac{\partial \rho^o}{\partial \rho} A^o_\rho + \rho^o \frac{\partial \varphi^o}{\partial \rho} A^o_\varphi + \frac{\partial z^o}{\partial \rho} A^o_z \\
A_\varphi &= \frac{1}{\rho}\frac{\partial \rho^o}{\partial \varphi} A^o_\rho + \frac{\rho^o}{\rho}\frac{\partial \varphi^o}{\partial \varphi} A^o_\varphi + \frac{1}{\rho}\frac{\partial z^o}{\partial \varphi} A^o_z \\
A_z &= \frac{\partial \rho^o}{\partial z} A^o_\rho + \rho^o \frac{\partial \varphi^o}{\partial z} A^o_\varphi + \frac{\partial z^o}{\partial z} A^o_z
\end{aligned} \tag{5.20}$$

In spherical polar coordinates, r, ϑ, φ we find for the field

$$\begin{aligned}
B_r &= \frac{\partial r}{\partial r^o} B^o_r + \frac{1}{r^o}\frac{\partial r}{\partial \vartheta^o} B^o_\vartheta + \frac{1}{r^o \sin\vartheta^o}\frac{\partial r}{\partial \varphi^o} B^o_\varphi \\
B_\vartheta &= r \frac{\partial \vartheta}{\partial r^o} B^o_r + \frac{r}{r^o}\frac{\partial \vartheta}{\partial \vartheta^o} B^o + \frac{r}{r^o \sin\vartheta^o}\frac{\partial \vartheta}{\partial \varphi^o} B^o_\varphi \\
B_\varphi &= r \sin\vartheta \frac{\partial \varphi}{\partial r^o} B^o_r + \frac{r \sin\vartheta}{r^o}\frac{\partial \varphi}{\partial \vartheta^o} B^o_\vartheta + \frac{r \sin\vartheta}{r^o \sin\vartheta^o}\frac{\partial \varphi}{\partial \varphi^o} B^o_\varphi
\end{aligned} \tag{5.21}$$

and for the vector potential

$$A_r = \frac{\partial r^o}{\partial r} A_r^o + r^o \frac{\partial \vartheta^o}{\partial r} A_\vartheta^o + r^o \sin\vartheta^o \frac{\partial \varphi^o}{\partial r} A_\varphi^o$$

$$A_\vartheta = \frac{1}{r} \frac{\partial r^o}{\partial \vartheta} A_r^o + \frac{r^o}{r} \frac{\partial \vartheta^o}{\partial \vartheta} A_\vartheta^o + \frac{r^o \sin\vartheta^o}{r} \frac{\partial \varphi^o}{\partial \vartheta} A_\varphi^o \qquad (5.22)$$

$$A_\varphi = \frac{1}{r \sin\vartheta} \frac{\partial r^o}{\partial \varphi} A_r^o + \frac{r^o}{r \sin\vartheta} \frac{\partial \vartheta^o}{\partial \varphi} A_\vartheta^o + \frac{r^o \sin\vartheta^o}{r \sin\vartheta} \frac{\partial \varphi^o}{\partial \varphi} A_\varphi^o$$

In the Lagrangian formulation of fluid motion we are concerned with the trajectories of the fluid particles,

$$\mathbf{r} = \mathbf{r}(\mathbf{r}^o, t) \qquad (5.23)$$

Given these three relationships we can compute the partial deviatives appearing in the preceding formulas. As a rule, however, we do not start from a set of trajectories but from a velocity field, $\mathbf{v}(\mathbf{r}, t)$, corresponding more closely to the Eulerian point of view. If the fluid motion is stationary we have $\partial \mathbf{v}/\partial t = 0$ and the trajectories (5.23) are the solutions of the differential equations

$$d\mathbf{r}/dt = \mathbf{v}(\mathbf{r}, \mathbf{r}^o) \qquad (5.24)$$

where it is assumed that the initial positions of the particles may appear as parameters.

We may inquire into cases in which (5.24) can be integrated by general methods. One such case is

$$dx/dt = v_x(x, x_i^o) \ , \quad dy/dt = v_y(y, x_i^o)$$
$$dz/dt = v_z(z, x_i^o) \qquad (5.25)$$

The integration can be carried out at once in the form

$$t = \int dx/v_x = \int dy/v_y = \int dz/v_z \qquad (5.26)$$

whence the trajectories (5.23) follow by inversion of the functional relationship between t and x, etc. Now the differential equations (5.25) are purely algebraic relationships with respect to the dependence of the x_i upon the x_i^o, hence these equations and their integrals may be applied to curvilinear coordinates without further complications. Among the solutions are in particular helical trajectories such as a cylindrical helix or a helix winding along a cone. Some of these motions will appear in our later analysis.

In order to get a clearer though somewhat elementary conception of amplificatory processes we next discuss the effects of induction for a field that is homogeneous over some region of space. By (5.17) the deformation of the field depends only on the components $\partial x_i/\partial x_k^o$ of the strain tensor. (This terminology agrees with the conventional definition of strain; the strains are here finite for finite times.) If, as usual, we disregard a pure rotation of the fluid as if solidified (in which case it is readily shown that the field rotates with the fluid) we are left with a symmetrical strain tensor which may be decomposed into a pure dilatation (expansion or compression) and a pure shear. Consider first a pure shearing strain. We may in this case assume the fluid to be incompressible. If we let the original field be in the z-direction and the fluid motion in the x-direction (5.17) reduces to

$$B_x = B_z^o(\partial x/\partial z^o) \ , \quad B_y = 0 \ , \quad B_z = B_z^o \qquad (5.27)$$

The field energy is

$$1/2\ B^2 = 1/2\ (B_z^o)^2\left[1 + (\partial x/\partial z^o)^2\right] \qquad (5.28)$$

The field increases linearly with the strain. We see that amplification of a field occurs whenever a velocity shear field is superposed perpendicularly upon an existing magnetic field. The amplification is linear in time for a stationary motion as appears if we write the first of (5.27) in the form

$$B_x = B_z^o\ t\ (\partial v_x/\partial z)$$

This is illustrated in Fig. 1 which shows on the left the velocity profile and on the right the "stretching" of the original (dashed) magnetic lines of force in the x-direction. In principle, an amplificatory process of this kind may be continued indefinitely, especially if the velocity field is circular (Sec. 6) rather than along straight lines.

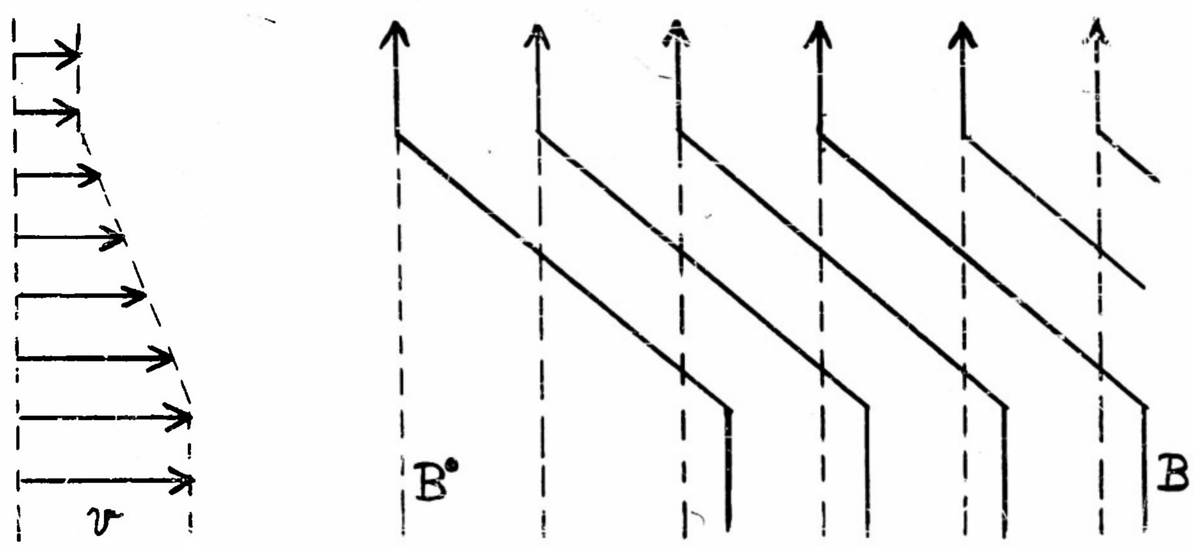

Figure 1

It is interesting to note that this simple process by (5.28) always gives an increase of the field energy. In order to see what happens to the energy in more general cases, let us return

to (5.17) and transform the stress tensor to diagonal form by a suitable rotation of the cartesian axes to which the x_i refer (we leave the axes to which the x_i^o refer unchanged). Let B_i now refer to the components of $\underset{\sim}{B}$ in the new axes (the B_i^o remaining unchanged). If ε_i are the elements of the diagonalized stress tensor the energy becomes

$$1/2 \sum_i B_i^2 = 1/2 \sum_i (B_i^o)^2 \varepsilon_i^2 \qquad (5.29)$$

whereas the condition of incompressibility (pure shear) merely requires $\sum \varepsilon_i = 0$. Now (5.29) can correspond to a decrease or an increase of the magnetic energy, depending on the magnitude of the ε_i, but for sufficiently large strains there will be an increase. This tendency of the magnetic energy toward increase is in the long run offset by the action of the ponderomotive forces (2.17) when they grow large as the field increases: By a well-known principle these forces act always in such a way that they in turn tend to decrease the magnetic field energy. (As pointed out already, the dynamo theory is not based upon a trend toward statistical equilibrium hereby implied, but on certain general dynamical principles which will be discussed later.)

Consider the case of a pure dilatation (expansion or contraction) in the absence of shear. Let B^o be in the z-direction and assume for simplicity that the stress tensor is diagonal along the cartesian axes given. Then (5.17) reduces to

$$B_z = (\rho/\rho^o) B_z^o \, (\partial z/\partial z^o) \qquad (5.30)$$

We may exemplify this by a gas cloud containing a homogeneous magnetic field (for instance a cloud of ionized gas shot out by

the sun) expanding into a vacuum. The total magnetic energy of the homogeneous cloud is proportional to B^2/ρ. If we let the cloud expand laterally, constraining the particles so that they cannot move in the z-direction, the magnetic energy diminishes proportional to ρ; if the cloud is prevented from lateral expansion and expands only along the z-axis, the magnetic energy goes as ρ^{-1}, thus it would increase on expansion; if the cloud expands laterally and contracts longitudinally (as suggested by the direction of the Maxwellian stresses) in such a way that the density remains unchanged, the magnetic energy changes as $(\Delta z)^2$ where Δz is the change in extension in the z-direction. For a gas, the internal thermal energy changes on expansion as $\rho^{\gamma-1}$ where γ is the ratio of the specific heats. Under suitable constraints the magnetic energy may thus increase at the expense of the thermal energy. We might remark that for a volume with a homogeneous magnetic field the ponderomotive forces (2.17) vanish on the inside; they appear as stresses on the boundary of the volume and are transmitted to the inside through hydrostatic pressures.

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